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Eater ideals in Jordan algebras

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Abstract

In this paper we study the relationship of several ideals of the free special Jordan algebra. In particular, we show that the ideal of hearty n-tad eaters coincides with that of imbedded n-tad eaters over an arbitrary ring of scalars. In the linear case, we show that they coincide with the ideal of n-tad eaters. The distance between the different eater submodules and their cores is also studied. © 1998 Elsevier Science B.V.

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0. Introduction

The ideals of n-tad eaters play a central role in the description of strongly prime linear Jordan algebras [6]. The extension of [6] to quadratic algebras [4] requires a combinatorial extra-effort in the form of new ideals of polynomials, namely, the so-called imbedded n-tad eaters and hearty eaters. A nonzero hermitian ideal of the free special Jordan algebra is obtained by McCrimmon and Zelmanov in [4] with the set of hearty pentad eaters, and several relations between the different sets of polynomials are established. D'Amour gives analogues of hearty eaters for Jordan triple systems in [1] and uses them in the study of strongly prime Jordan triple systems [2].

Our aim is to further investigate the relationship between n-tad eaters, imbedded n-tad eaters and hearty n-tad eaters, as well as to study the distance between the submodules consisting of these polynomials and the biggest ideals (the cores) contained in them. With purely combinatorial techniques some equalities relating associative and usual n-tads are given in Section 1. This allows us to show that the set E_n of n-tad eaters is always an outer ideal when n is odd (1.6) and to study in (1.9) the

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distance between consecutive E_n, E_{n+1} and their cores, extending some of the results of [4]. In Section 2, imbedded n-tad eaters are studied with the help of the combinatorial lemmas of the previous section. Namely, it is proved that the set of imbedded n-tad eaters IE_n is always an ideal when n is odd (2.7), as well as an analogue (2.8) of (1.9). To simplify the argument we first show that imbedded n-tad eaters are exactly those polynomials which eat associative n-tads (2.3), a fact which is also used to study the connection between E_n and IE_n. Precedents of these results which have inspired a part of this work can be found in [5, pp. 69–70]. In Section 3, the equality between imbedded n-tad eaters and hearty n-tad eaters is established (3.6). Given an arbitrary adic family F on the free special Jordan algebra, a realization of the free associative algebra can be built (3.4), so that calculations with F can be reduced to associative n-tads. The use of this model also justifies the simplified definition of adic family on the free special Jordan algebra given in (3.1).

0.1. Preliminaries

Throughout this paper we will deal with an arbitrary ring of scalars Φ. Unless explicitly said, the existence of $\frac{1}{2} \in \Phi$ is not assumed. Our main reference for basic results and terminology will be [4]. To make the text as self-contained as possible, we will recall some basic facts.

0.1. A *unital Jordan algebra* over Φ consists of a Φ-module J, a distinguished element $1 \in J$, and a quadratic map $U : J \to End_\Phi(J)$ such that

$$U_1 = Id, \qquad U_x V_{y,x} = V_{x,y} U_x = U_{U_x y, x}, \qquad U_{U_x y} = U_x U_y U_x$$

hold in all scalar extensions, where

$$V_{x,y} z = \{xyz\} = U_{x,z} y \quad (U_{x,z} = U_{x+z} - U_x - U_z).$$

A *Jordan algebra* is just a subspace $J = (J, U, (\ )^2)$ of some unital Jordan algebra closed under the products $U_x y$ and the *square*

$$x^2 = U_x 1.$$

If $\frac{1}{2} \in \Phi$ we can characterize these algebras axiomatically as the *linear Jordan algebras* with product $x \cdot y = \frac{1}{2} U_{x,y} 1$ satisfying

$$x \cdot y = y \cdot x, \qquad (x^2 \cdot y) \cdot x = x^2 \cdot (y \cdot x).$$

Any associative algebra A gives rise to a Jordan algebra A^+ via

$$U_x y := xyx, \qquad x^2 = xx.$$

A Jordan algebra is *special* if it is isomorphic to a Jordan subalgebra of some A^+. An important example arises from an associative algebra A with an involution $*$ by considering the *hermitian algebra* $H(A, *)$ of all $*$-symmetric elements in A.

0.2. The free (unital) associative Φ-algebra over an infinite set X of variables will be denoted by $\mathrm{Ass}(X)$ and its elements will be called *associative polynomials*. Inside $\mathrm{Ass}(X)$, the free special (unital) Jordan algebra over X, i.e., the Jordan subalgebra of $\mathrm{Ass}(X)^{(+)}$ generated by X (and 1), will be denoted by $\mathrm{SJ}(X)$ and its elements will be called *Jordan polynomials*. An associative or Jordan polynomial p will be written $p(y_1, \ldots, y_m)$ if only the variables $y_1, \ldots, y_m \in X$ are involved in p. In $\mathrm{Ass}(X)$ one can consider the so-called standard involution $*$, which fixes the elements of X. Jordan polynomials are always symmetric with respect to $*$:

$$\mathrm{SJ}(X) \subseteq H(\mathrm{Ass}(X), *) \subseteq \mathrm{Ass}(X),$$

where $H(\mathrm{Ass}(X), *)$ denotes the set of $*$-symmetric elements in $\mathrm{Ass}(X)$ [4, p. 144].

Notice that $\mathrm{Ass}(X)$ and $\mathrm{SJ}(X)$ are just the free unital hulls of their corresponding non-unital analogues, so that imposing the existence of unit elements is not a real restriction: any map $X \to J$, where J is a not necessarily unital Jordan algebra, can be extended to a unique (unital) algebra homomorphism $\mathrm{SJ}(X) \to \Phi 1 \oplus J$, where $\Phi 1 \oplus J$ is the free unital hull of J.

0.3. The *trace* function on $\mathrm{Ass}(X)$ is defined by $\{a\} := a + a^*$ for any element $a \in \mathrm{Ass}(X)$. Notice that for Jordan polynomials $a_1, \ldots, a_n$ (indeed for arbitrary symmetric polynomials) the equality

$$\{a_1 \ldots a_n\} = a_1 \ldots a_n + a_n \ldots a_1$$

holds. Polynomials of the form $\{a_1 \ldots a_n\}$, where $a_1, \ldots, a_n \in \mathrm{SJ}(X)$, will be called *n-tads*. All n-tads are symmetric polynomials and, if $n \leq 3$ they are Jordan polynomials [4, p. 144]. The associative polynomials

$$a_1 \ldots a_n,$$

$a_1, \ldots, a_n \in \mathrm{SJ}(X)$, will be called *associative n-tads* [4, p. 188].

0.4. We can generate $\mathrm{SJ}(X)$ as a Φ-module with *Jordan monomials*, which are defined inductively from the variables by Jordan products: the unit and all elements in X are Jordan monomials and, given Jordan monomials a, b, c, the products

$$U_a b, \qquad U_{a,c} b \quad (a^2 = U_a 1, \quad a \circ b = U_{a,b} 1)$$

are also Jordan monomials. Unlike in the associative case, the set of Jordan monomials is not a basis of $\mathrm{SJ}(X)$ (e.g. $2x^2 = x \circ x$). A Jordan monomial p is a homogeneous associative polynomial and so its degree, denoted by $\deg p$, can be considered.

0.5. Recall that an outer ideal L of a (not necessarily unital) Jordan algebra J is a submodule of J such that $U_J L + J \circ L \subseteq L$ [4, 0.13]. In the linear case ($\frac{1}{2} \in \Phi$) outer ideals are just ideals. In general, if L is an outer ideal of J then $2L$ is an ideal of J: for any $x \in L$,

$$(2x)^2 = 4x^2 = 2(x \circ x) \in 2L,$$

$$U_{2x}y = 4U_x y = 2\{xyx\} = 2(\{xyx\} - \{yxx\} + \{yxx\})$$
$$= 2((x \circ y) \circ x - (x \circ x) \circ y + \{yxx\}) \in 2L.$$

1. *n*-tad eaters

1.1. A Jordan polynomial $p(y_1, \ldots, y_m)$ is called an *n-tad eater* if

$$\{x_1 \ldots x_{n-1} p(y_1, \ldots, y_m)\} = q(x_1, \ldots, x_{n-1}, y_1, \ldots, y_m)$$

for some $q(x_1, \ldots, x_{n-1}, y_1, \ldots, y_m) \in \mathrm{SJ}(X)$, $x_1, \ldots, x_{n-1} \in X \backslash \{y_1, \ldots, y_m\}$. We can replace variables by arbitrary elements of $\mathrm{SJ}(X)$, so that $p \in \mathrm{SJ}(X)$ is an n-tad eater if and only if

$$\{\overbrace{\mathrm{SJ}(X) \ldots \mathrm{SJ}(X)}^{n-1 \text{ factors}} p\} \subseteq \mathrm{SJ}(X)$$

[4, 12.1].

1.2. The set of all n-tad eaters is denoted by E_n. It is a Φ-submodule of $\mathrm{SJ}(X)$. Replacing variables by unit elements gives the chain of containments

$$\mathrm{SJ}(X) = E_1 = E_2 = E_3 \supseteq E_4 \supseteq E_5 \supseteq \ldots . \tag{1}$$

The core of E_n, i.e. the biggest ideal of $\mathrm{SJ}(X)$ contained in E_n, will be denoted by T_n. The ideals T_n satisfy

$$\mathrm{SJ}(X) = T_1 = T_2 = T_3 \supseteq T_4 \supseteq T_5 \supseteq \ldots . \tag{2}$$

Both E_n, T_n are invariant under all linearizations and under all homomorphisms of $\mathrm{SJ}(X)$ [4, p. 183].

1.3. We recall [4, 12.14] that an n-tad eater eats n-tads no matter where it occurs,

$$p \in E_n \Rightarrow \{\overbrace{\mathrm{SJ}(X) \ldots \mathrm{SJ}(X) p \mathrm{SJ}(X) \ldots \mathrm{SJ}(X)}^{n \text{ factors}}\} \subseteq \mathrm{SJ}(X).$$

Next, we introduce two associative polynomials which will be important tools in the sequel. For any $x, x_1, \ldots, x_n \in X$, we define the *walking polynomial*

$$W_x(x_1, \ldots, x_n) := xx_1 \ldots x_n + (-1)^{n-1} x_1 \ldots x_n x.$$

If n is even we can also define the *running polynomial*

$$R_x(x_1, \ldots, x_n) := xx_1 \ldots x_n + (-1)^{(n/2)-1} x_2x_1x_4x_3 \ldots x_{2i}x_{2i-1} \ldots x_nx_{n-1}x.$$

The next lemma shows the kind of "steps" out of which walking and running is made.

1.4. Lemma. *Let* $x, x_1, \ldots, x_n \in X$.

(i) $W_x(x_1, \ldots, x_n) = \sum_{i=1}^{n} (-1)^{i-1} x_1 \ldots x_{i-1}\{xx_i\}x_{i+1} \ldots x_n$.

(ii) $R_x(x_1, \ldots, x_n) = \sum_{i=1}^{n/2} (-1)^{i-1} x_2x_1 \ldots x_{2i-2}x_{2i-3}\{xx_{2i-1}x_{2i}\}x_{2i+1} \ldots x_n$ (*n even*).

Proof. (i) The equality is clear for $n = 1$. Let $n \geq 2$ and assume that the assertion is true for $n - 1$. Now

$$\begin{aligned} W_x(x_1, \ldots, x_n) &= xx_1 \ldots x_n + (-1)^{n-1} x_1 \ldots x_n x \\ &= (xx_1 \ldots x_{n-1} + (-1)^{n-2} x_1 \ldots x_{n-1}x)x_n \\ &\quad + (-1)^{n-1}(x_1 \ldots x_{n-1}xx_n + x_1 \ldots x_{n-1}x_nx) \\ &= W_x(x_1, \ldots, x_{n-1})x_n + (-1)^{n-1} x_1 \ldots x_{n-1}\{xx_n\}. \end{aligned}$$

By the induction assumption, the last term equals

$$\left(\sum_{i=1}^{n-1} (-1)^{i-1} x_1 \ldots x_{i-1}\{xx_i\}x_{i+1} \ldots x_{n-1}\right)x_n + (-1)^{n-1} x_1 \ldots x_{n-1}\{xx_n\}$$
$$= \sum_{i=1}^{n} (-1)^{i-1} x_1 \ldots x_{i-1}\{xx_i\}x_{i+1} \ldots x_n.$$

(ii) Let $n = 2m$, $m \geq 2$. The equality

$$R_x(x_1, \ldots, x_n) = R_x(x_1, \ldots, x_{n-2})x_{n-1}x_n + (-1)^{m-1} x_2x_1 \ldots x_{n-2}x_{n-3}\{xx_{n-1}x_n\}$$

follows directly from the definition of the running polynomial. Now (ii) follows by induction on m since the case $m = 1$ is obvious. □

1.5. Lemma. *If* $x_1, \ldots, x_k, y, z \in X$, *where* k *is an even positive integer then*

$$x_1 \ldots x_k zyz - x_1\{x_2 \ldots x_k zy\}z - (-1)^{k/2} x_1 y x_{k-1}x_k \ldots x_3x_4zx_2z$$

lies in the linear span in Ass(X) *of the elements*

$$x_1 y x_{\sigma(2)} \ldots x_{\sigma(i-1)}\{zx_{\sigma(i)}x_{\sigma(i+1)}\}x_{\sigma(i+2)} \ldots x_{\sigma(k)}z,$$

where σ *is a permutation of* $\{2, \ldots, k\}$.

Proof. Recall that, by definition of the running polynomial, we have

$$x_1 y R_z(x_k, \ldots, x_3)x_2z = x_1 y[zx_kx_{k-1} \ldots x_3 + (-1)^{(k/2)-2} x_{k-1}x_k \ldots x_3x_4z]x_2z.$$

and then

$$\begin{aligned} x_1 \ldots x_k zyz &= x_1\{x_2 \ldots x_k zy\}z - x_1 yzx_k \ldots x_2 z \\ &= x_1\{x_2 \ldots x_k zy\}z - x_1 yR_z(x_k, \ldots, x_3)x_2 z \\ &\quad + (-1)^{k/2} x_1 yx_{k-1}x_k \ldots x_3 x_4 zx_2 z. \end{aligned}$$

Now the result follows from (1.4) (ii) since $x_1 yR_z(x_k, \ldots, x_3)x_2 z$ is in the linear span of the elements

$$x_1 yx_{\sigma(2)} \ldots x_{\sigma(i-1)}\{zx_{\sigma(i)}x_{\sigma(i+1)}\}x_{\sigma(i+2)} \ldots x_{\sigma(k)}z,$$

where σ is a permutation of $\{2, \ldots, k\}$. □

We will use the previous result in the next theorem which extends to an arbitrary odd n [4, 12.5 (ii)].

1.6. Theorem. *Let n be an odd positive integer.*

(i) *E_n is an outer ideal of* SJ(X) *and $2E_n \subseteq T_n$.*

(ii) *If $\frac{1}{2} \in \Phi$ then $E_n = T_n$.*

Proof. (i) We know from (1.2) that $E_1 = T_1$. We will show that E_{k+1} is an outer ideal of SJ(X) for every even k. Let $p \in E_{k+1}$, $a_1, \ldots, a_k, b \in$ SJ(X). By taking traces in (1.5), using $p \in E_{k+1}$ and (1.3) we obtain

$$\{a_1 \ldots a_k bpb\} - \{a_1\{a_2 \ldots a_k bp\}b\} - (-1)^{k/2}\{a_1 pa_{k-1}a_k \ldots a_3 a_4 ba_2 b\} \in \mathrm{SJ}(X).$$

Moreover,

$$\begin{aligned} &\{a_1\{a_2 \ldots a_k bp\}b\} \pm \{a_1 pa_{k-1}a_k \ldots a_3 a_4 ba_2 b\} \\ &\quad = \{a_1\{a_2 \ldots a_k bp\}b\} \pm \{a_1 pa_{k-1}a_k \ldots a_3 a_4 (U_b a_2)\} \\ &\quad \in \{\mathrm{SJ}(X)\mathrm{SJ}(X)\mathrm{SJ}(X)\} \pm \{\overbrace{\mathrm{SJ}(X)p\mathrm{SJ}(X) \ldots \mathrm{SJ}(X)}^{k+1 \text{ factors}}\} \subseteq \mathrm{SJ}(X) \end{aligned}$$

by (1.3) since $p \in E_{k+1}$ and we have shown $U_b p = bpb \in E_{k+1}$. This proves that E_n is an outer ideal of SJ(X) since SJ(X) is unital. Now $2E_n \subseteq T_n$ since $2E_n$ is an ideal of SJ(X) by (0.5).

(ii) Follows clearly from (i). □

Some reverse inclusions of (1.2) are given in [4, 12.5] while in the linear case the equalities $T_4 = T_5 = T_6 = T_7$ are part of the Jordan folklore. Our next results are aimed at strengthening and unifying the previous assertions.

1.7. Lemma. *For any integer $n \geq 2$, $x_1, \ldots, x_n \in X$,*

$$\sum_{i=2}^{n} (-1)^{(i-1)(i+2)/2} x_n \ldots x_{i+1} W_{x_i}(x_1, \ldots, x_{i-1}) = x_n \ldots x_1 - (-1)^{(n-1)n/2} x_1 \ldots x_n.$$

Proof. We will carry out an induction on n. For $n = 2$ our assertion is just the definition of the walking polynomial $W_{x_2}(x_1) = x_2x_1 + x_1x_2$. Let us assume the equality for some $n \geq 2$. Now

$$
\begin{aligned}
&\sum_{i=2}^{n+1} (-1)^{(i-1)(i+2)/2} x_{n+1} \dots x_{i+1} W_{x_i}(x_1, \dots, x_{i-1}) \\
&= x_{n+1}\left(\sum_{i=2}^{n} (-1)^{(i-1)(i+2)/2} x_n \dots x_{i+1} W_{x_i}(x_1, \dots, x_{i-1}) \right) \\
&\quad + (-1)^{n(n+3)/2} W_{x_{n+1}}(x_1, \dots, x_n) \\
&= x_{n+1}(x_n \dots x_1 - (-1)^{\frac{1}{2}(n-1)n} x_1 \dots x_n) + (-1)^{n(n+3)/2}(x_{n+1}x_1 \dots x_n \\
&\quad + (-1)^{n-1} x_1 \dots x_{n+1}) \quad \text{(by the induction assumption)} \\
&= x_{n+1}x_n \dots x_1 + (-1)^{n(n+3)/2+n-1} x_1 \dots x_{n+1} \quad (\text{since } \tfrac{1}{2}n(n+3) \\
&\qquad\qquad - \tfrac{1}{2}(n-1)n = \tfrac{1}{2}(n^2 + 3n - (n^2 - n)) = 2n) \\
&= x_{n+1} \dots x_1 + (-1)^{(n^2+5n-2)/2} x_1 \dots x_{n+1} \\
&= x_{n+1} \dots x_1 - (-1)^{(n^2+n)/2} x_1 \dots x_{n+1} \quad (\text{since } \tfrac{1}{2}(n^2 + 5n - 2) \\
&\qquad\qquad - \tfrac{1}{2}(n^2 + n) = \tfrac{1}{2}(4n - 2) = 2n - 1). \qquad \square
\end{aligned}
$$

Given an associative n-tad $a_1 \dots a_n$, for $a_1, \dots, a_n \in \mathrm{SJ}(X)$, the polynomial

$$a_{\sigma(1)} \dots \{a_{\sigma(i)}a_{\sigma(i+1)}\} \dots a_{\sigma(n)}$$

obtained by a permutation σ of the indexes and a Jordan product $a_{\sigma(i)} \circ a_{\sigma(i+1)} = \{a_{\sigma(i)}a_{\sigma(i+1)}\}$ will be called a *reduction* of $a_1 \dots a_n$.

1.8. Proposition. *Let n be a positive integer, $x_1, \dots, x_n \in X$.*

(i) *If $n = 4k$ or $n = 4k + 1$ for some integer k, then $2x_1 \dots x_n - \{x_1 \dots x_n\}$ is a linear combination with coefficients ± 1 of reductions of $x_1 \dots x_n$.*

(ii) *If $n = 4k + 2$ or $n = 4k + 3$ for some integer k, then $\{x_1 \dots x_n\}$ is a linear combination with coefficients ± 1 of reductions of $x_1 \dots x_n$.*

Proof. Notice that $n = 4k$ or $n = 4k + 1$ if and only if $(-1)^{1/2(n-1)n} = 1$ and $n = 4k + 2$ or $n = 4k + 3$ if and only if $(-1)^{1/2(n-1)n} = -1$. Now, apply (1.7) and (1.4) (i). $\square$

1.9. Theorem. *Let k be a positive integer. Then*

(i) $2T_{4k} \subseteq E_{4k+1}$, $2T_{4k+1} \subseteq E_{4k+2}$ *and* $2T_{4k+2} \subseteq E_{4k+3}$.

(ii) *If $\frac{1}{2} \in \Phi$ then $T_{4k} = T_{4k+1} = T_{4k+2} = T_{4k+3}$.*

Proof. (i) If $n = 4k$ or $n = 4k + 1$ then

$$2x_0x_1 \dots x_n - x_0\{x_1 \dots x_n\} = x_0(2x_1 \dots x_n - \{x_1 \dots x_n\})$$

is a linear combination of reductions of $x_0 \dots x_n$ by (1.8) (i). Putting $p \in T_n$ instead of x_n, evaluating $x_i \mapsto a_i \in \mathrm{SJ}(X)$ $(i = 0, \dots, n-1)$ and taking traces yields by (0.3) that $\{a_0 \dots a_{n-1} 2p\}$ is $\{a_0\{a_1 \dots a_{n-1}p\}\}$ plus a linear combination of traces of reductions of $a_0 \dots a_{n-1}p$, which are Jordan polynomials since $p \in T_n$ and hence $\{a_i p\} \in T_n$ for any a_i. Thus $2p \in E_{n+1}$.

If $n = 4k+2$ then $\{x_1 \dots x_{n+1}\}$ is a linear combination of reduction of $x_1 \dots x_{n+1}$. Thus any evaluation $x_i \mapsto a_i \in \mathrm{SJ}(X)(i = 1, \dots, n)$, $x_{n+1} \mapsto p \in T_n$ is a linear combination of reductions of $a_1 \dots a_n p$. By (0.3), taking traces gives that

$$\{a_1 \dots a_n 2p\} = 2\{a_1 \dots a_n p\} = \{\{a_1 \dots a_n p\}\}$$

is a linear combination of traces of reductions of $a_1 \dots a_n p$, which are Jordan polynomials since $p \in T_n$ implies $\{a_i p\} \in T_n$ as above.

(ii) Recall that T_{4k} is an ideal of $\mathrm{SJ}(X)$, which is contained in E_{4k+1} by (i) if $\frac{1}{2} \in \Phi$. Hence $T_{4k} \subseteq T_{4k+1}$. Similarly, $T_{4k+1} \subseteq T_{4k+2} \subseteq T_{4k+3}$. But $T_{4k+3} \subseteq T_{4k}$ by (1.2) (2). □

1.10. Corollary. *If* $\frac{1}{2} \in \Phi$, *then*

(i) $T_{4k} = T_{4k+1} = T_{4k+2} = T_{4k+3} = E_{4k+1} = E_{4k+2} = E_{4k+3}$ *for any positive integer* k. *Notice the equalities in the chains given in* (1.2)

$$\cdots \begin{array}{ccccccc} \subseteq T_{4k} & = & T_{4k+1} & = & T_{4k+2} & = & T_{4k+3} \subseteq \\ \cup| & & \| & & \| & & \| \\ \subseteq E_{4k} & \subseteq & E_{4k+1} & = & E_{4k+2} & = & E_{4k+3} \subseteq \end{array} \cdots,$$

(ii) $E_{n+1} \subseteq T_n$ *for any positive integer* n.

Proof. (i) follows from (1.9) and (1.6). Indeed, $E_{4k+3} \subseteq E_{4k+2} \subseteq E_{4k+1}$ by (1.2)(1), but $E_{4k+1} = T_{4k+1}$, $E_{4k+3} = T_{4k+3}$ by (.16)(ii), and $T_{4k} = T_{4k+1} = T_{4k+2} = T_{4k+3}$ by (1.9) (ii).

(ii) The cases $n = 4k, 4k+1, 4k+2$ follow from (i). If $n = 4k+3$ then $E_n = T_n$ by (1.6) and $E_{n+1} \subseteq E_n$ by (12)(1). □

1.11. Remarks. (i) A result analogous to Lemma 1.7 can be obtained for the running polynomial: For any positive integer m, $x_1, \dots, x_{2m+1} \in X$,

$$\begin{aligned} &\sum_{i=1}^{m} (-1)^{i-1} R_{x_{2i-1}}(x_{2i}, x_{2i+1}, \dots, x_{2m+1}) x_{2i-2} x_{2i-3} \dots x_2 x_1 \\ &\quad + (-1)^m \sum_{j=1}^{m-1} x_{2j+1} R_{x_{2j}}(x_{2j+3}, x_{2j+2}, x_{2j+5}, x_{2j+4}, \dots, x_{2m+1}, x_{2m}) \\ &\quad \times x_{2j-1} x_{2j-2} \dots x_2 x_1 \\ &= x_1 \dots x_{2m+1} + (-1)^{m-1} x_{2m+1} \dots x_1. \end{aligned}$$

If we call a 2-*reduction* of a given associative n-tad $a_1 \dots a_n (n \geq 3, a_1, \dots, a_n$ in $\mathrm{SJ}(X))$ the polynomial $a_{\sigma(1)} \dots \{a_{\sigma(i)} a_{\sigma(i+1)} a_{\sigma(i+2)}\} \dots a_{\sigma(n)}$ obtained from a permutation

σ of the indexes and a ternary Jordan product $\{a_{\sigma(i)}a_{\sigma(i+1)}\, a_{\sigma(i+2)}\}$, then the following partial improvement of Proposition 1.8 can be obtained from the above formula:

(i′) *If* $n = 4k + 1$ *for some integer k, then* $2x_1 \ldots x_n - \{x_1 \ldots x_n\}$ *is a linear combination with coefficients* ± 1 *of 2-reductions of* $x_1 \ldots x_n$.

(ii′) *If* $n = 4k + 3$ *for some integer k, then* $\{x_1 \ldots x_n\}$ *is a linear combination with coefficients* ± 1 *of 2-reductions of* $x_1 \ldots x_n$.

(ii) In the proof of (1.9)(i), the case $n = 4k + 2$ can also be proved by applying (1.8)(i) (or even the above (i′)) to $2x_2 \ldots x_n - \{x_2 \ldots x_n\}$ and multiplying by x_0x_1, which yields that $2x_0 \ldots x_n - x_0x_1\{x_2 \ldots x_n\}$ is a linear combination of reductions of $x_0 \ldots x_n$ and then proceed as in the case $n = 4k$ or $n = 4k + 1$.

(iii) Neither (1.8)(i), (ii) nor its partial improvements (i′), (ii′) can be used to obtain an analogue of (1.9)(i) for $n = 4k + 3$.

2. Imbedded *n*-tad eaters

2.1. A Jordan polynomial $p(y_1, \ldots, y_m) \in \mathrm{SJ}(X)$, $y_1, \ldots, y_m \in X$ is called an *imbedded n-tad eater* if

$$\{z_1 \ldots z_r x_1 \ldots x_{n-1} p u_1 \ldots u_s\} = \sum_{i=1}^{k} \{z_1 \ldots z_r p_1^i p_2^i p_3^i u_1 \ldots u_s\},$$

where $p_j^i(x_1, \ldots, x_{n-1}, y_1, \ldots, y_m) \in \mathrm{SJ}(X)$, for arbitrary positive integers r, s and $z_1, \ldots, z_r, x_1, \ldots, x_{n-1}, u_1, \ldots, u_s \in X$.

2.2. A Jordan polynomial $p(y_1, \ldots, y_m) \in \mathrm{SJ}(X)$, $y_1, \ldots, y_m \in X$ is called an *associative n-tad eater* if

$$x_1 \ldots x_{n-1} p = \sum_{i=1}^{k} p_1^i p_2^i p_3^i,$$

where $p_j^i(x_1, \ldots, x_{n-1}, y_1, \ldots, y_m) \in \mathrm{SJ}(X)$ for arbitrary $x_1, \ldots, x_{n-1} \in X$. By using the universal property of $\mathrm{Ass}(X)$, a Jordan polynomial $p(y_1, \ldots, y_m)$ is an associative n-tad eater if

$$\overbrace{\mathrm{SJ}(X) \ldots \mathrm{SJ}(X)}^{n \text{ factors}} p \subseteq \mathrm{SJ}(X)\mathrm{SJ}(X)\mathrm{SJ}(X).$$

The next result shows that the notions defined in (2.1) and (2.2) coincide.

2.3. Proposition. *A Jordan polynomial* $p(y_1, \ldots, y_m) \in \mathrm{SJ}(X)$, $y_1, \ldots, y_m \in X$ *is an associative n-tad eater if and only if it is an imbedded n-tad eater.*

Proof. It is clear from the definition that associative n-tad eaters are imbedded n-tad eaters. We will show the converse. Let $a, b, x_1, \ldots, x_{n-1} \in X \backslash \{y_1, \ldots, y_m\}$,

$a \neq b$, $a, b \notin \{x_1, \ldots, x_{n-1}\}$ and assume that $p(y_1, \ldots, y_m)$ is an imbedded n-tad eater. Hence

$$\{ax_1 \ldots x_{n-1}pb\} = \sum_{i=1}^{k} \{ap_1^i p_2^i p_3^i b\},$$

where $p_j^i(x_1, \ldots, x_{n-1}, y_1, \ldots, y_m) \in \mathrm{SJ}(X)$. Comparing in the previous equality the associative monomials beginning with a yields

$$ax_1 \ldots x_{n-1}pb = \sum_{i=1}^{k} ap_1^i p_2^i p_3^i b.$$

Thus $x_1 \ldots x_{n-1}p = \sum_{i=1}^{k} p_1^i p_2^i p_3^i$ and p is an associative n-tad eater. $\square$

2.4. The set of imbedded n-tad eaters is a submodule of $\mathrm{SJ}(X)$, denoted by IE_n, whose core will be denoted by I_n. As for n-tad eaters the following chains of containments hold

$$\mathrm{SJ}(X) = IE_1 = IE_2 = IE_3 \supseteq IE_4 \supseteq IE_5 \supseteq \ldots, \tag{1}$$

$$\mathrm{SJ}(X) = I_1 = I_2 = I_3 \supseteq I_4 \supseteq I_5 \supseteq \ldots. \tag{2}$$

We also have the obvious relation between n-tad eaters and imbedded n-tad eaters

$$IE_n \subseteq E_n, \qquad I_n \subseteq T_n, \tag{3}$$

for all n (cf. [4, 13.1, 13.2]).

2.5. It is not known whether an element $p \in IE_n$ eats imbedded n-tads (equivalently, with a proof like the one in (2.3), associative n-tads) from any position. But this is true if $p \in I_n$:

$$\overbrace{\mathrm{SJ}(X) \ldots \mathrm{SJ}(X)p\mathrm{SJ}(X) \ldots \mathrm{SJ}(X)}^{n \text{ factors}} \subseteq \mathrm{SJ}(X)\mathrm{SJ}(X)\mathrm{SJ}(X). \tag{1}$$

We remark that the above property holds for any p lying in an outer ideal B of $\mathrm{SJ}(X)$ contained in IE_n (cf. [4, 12.14]): If

$$a_1 \ldots a_r p a_{r+2} \ldots a_n \in \mathrm{SJ}(X)\mathrm{SJ}(X)\mathrm{SJ}(X)$$

for any $a_1, \ldots, a_r, a_{r+2}, \ldots, a_n \in \mathrm{SJ}(X)$ and any $p \in B$, then

$$\begin{aligned} a_1 \ldots a_{r-1} p a_r a_{r+2} \ldots a_n &= a_1 \ldots a_{r-1}(a_r \circ p)a_{r+2} \ldots a_n - a_1 \ldots a_r p a_{r+2} \ldots a_n \\ &= 1a_1 \cdots a_{r-1}(a_r \circ p)a_{r+2} \ldots a_n - a_1 \ldots a_r p a_{r+2} \ldots a_n \in \mathrm{SJ}(X)\mathrm{SJ}(X)\mathrm{SJ}(X) \end{aligned}$$

since $a_r \circ p \in B$.

Anyway, for an arbitrary $p \in IE_n$, p eats associative n-tads from positions' numbers 1, 2, 3, $n-2$, $n-1$ and n:

$$a_1 \ldots a_{n-1}p,\ a_1 \ldots a_{n-2}pa_{n-1},\ a_1 \ldots a_{n-3}pa_{n-2}a_{n-1} \in \mathrm{SJ}(X)\mathrm{SJ}(X)\mathrm{SJ}(X), \tag{2}$$

$$pa_1 \ldots a_{n-1},\ a_1pa_2 \ldots a_{n-1},\ a_1a_2pa_3 \ldots a_{n-1} \in \mathrm{SJ}(X)\mathrm{SJ}(X)\mathrm{SJ}(X), \tag{3}$$

for any $a_1, \dots, a_{n-1} \in \mathrm{SJ}(X)$. Indeed $a_1 \dots a_{n-1} p \in \mathrm{SJ}(X)\mathrm{SJ}(X)\mathrm{SJ}(X)$ by (2.3). Now

$$a_1 \dots a_{n-2} p a_{n-1} = \{a_1 \dots a_{n-2} p\} a_{n-1} - \{p a_{n-2} \dots a_1 a_{n-1}\} + a_{n-1} a_1 \dots a_{n-2} p \in$$

$$\mathrm{SJ}(X)\mathrm{SJ}(X) + \mathrm{SJ}(X) + \mathrm{SJ}(X)\mathrm{SJ}(X)\mathrm{SJ}(X) \subseteq \mathrm{SJ}(X)\mathrm{SJ}(X)\mathrm{SJ}(X)$$

by (1.3) since $p \in IE_n \subseteq E_n \subseteq E_{n-1}$. Similarly, $a_1 \dots a_{n-3} p a_{n-2} a_{n-1}$ lies in $\mathrm{SJ}(X)\mathrm{SJ}(X)\mathrm{SJ}(X)$, and (3) follows from (2) by applying the standard involution.

In the next result we study the converse of (2.4)(3).

2.6. Theorem. *Let $n \geq 3$ be a positive integer.*

(i) $2^{n-3} E_n \subseteq IE_n$, $2^{n-3} T_n \subseteq I_n$.

(ii) *If $\frac{1}{2} \in \Phi$ then $E_n = IE_n$, $T_n = I_n$.*

Proof. (i) By (1.2) and (2.4) the result is clear for $n = 3$. Now we will carry out an induction on n. Let $n \geq 4$ and assume (i) for indexes less than n. Let $a_1, \dots, a_{n-1} \in \mathrm{SJ}(X)$, $p \in E_n$. Assume first $n = 4k$ or $n = 4k + 1$ for some integer k. By (1.8)(i)

$$a_1 \dots a_{n-1} 2^{n-3} p - \{a_1 \dots a_{n-1} 2^{n-4} p\}$$

is a linear combination of reductions of $a_1 \dots a_{n-1} 2^{n-4} p$. Any such reduction is an associative $(n-1)$-tad containing either $2^{n-4} p$ or $\{a_i 2^{n-4} p\} = 2^{n-4}\{a_i p\}$. If $n = 4k$ then, by (1.6) and the induction assumption, $2^{n-4} E_{n-1}$ is an outer ideal contained in IE_{n-1}, and $2^{n-4} p, \{a_i 2^{n-4} p\} \in 2^{n-4} E_{n-1}$. If $n = 4k + 1$ then $2^{n-4} E_n$ is an outer ideal by (1.6), $2^{n-4} p, \{a_i 2^{n-4} p\} \in 2^{n-4} E_n$ and $2^{n-4} E_n \subseteq 2^{n-4} E_{n-1} \subseteq IE_{n-1}$ by the induction assumption. By the remark following (2.5)(1), $2^{n-4} p, \{a_i 2^{n-4} p\}$ eat associative $(n-1)$-tads from any position and the above-mentioned reductions lie in $\mathrm{SJ}(X)\mathrm{SJ}(X)\mathrm{SJ}(X)$. Now, the n-tad $\{a_1 \dots a_{n-1} 2^{n-4} p\}$ lies in $\mathrm{SJ}(X)\mathrm{SJ}(X)\mathrm{SJ}(X)$ since $2^{n-4} p \in E_n$, hence

$$a_1 \dots a_{n-1} 2^{n-3} p \in \mathrm{SJ}(X)\mathrm{SJ}(X)\mathrm{SJ}(X),$$

showing $2^{n-3} p \in IE_n$ (by (2.3)).

The cases $n = 4k + 2$ and $n = 4k + 3$ follow analogously by applying (1.8)(i) to

$$a_3 \dots a_{n-1} 2^{n-3} p - \{a_3 \dots a_{n-1} 2^{n-4} p\}$$

to show that

$$a_1 \dots a_{n-1} 2^{n-3} p - a_1 a_2 \{a_3 \dots a_{n-1} 2^{n-4} p\}$$

is a linear combination of reductions of $a_1 \dots a_{n-1} 2^{n-4} p$, and noticing that

$$a_1 a_2 \{a_3 \dots a_{n-1} 2^{n-4} p\} \in \mathrm{SJ}(X)\mathrm{SJ}(X)\mathrm{SJ}(X)$$

since $\{a_3 \dots a_{n-1} 2^{n-4} p\} \in \mathrm{SJ}(X)$ by $2^{n-4} p \in E_n \subseteq E_{n-2}$.

(ii) Is straightforward. □

The analogue of (1.6) is even stronger for imbedded n-tad eaters.

2.7. Theorem. *For any positive integer $n \neq 4$, IE_n is an inner ideal of* SJ(X). *Moreover, if n is odd then IE_n is an ideal of* SJ(X), *i.e.,* $IE_n = I_n$.

Proof. We will show that IE_n is an inner ideal if $n \neq 4$. The result is known for $n = 1$, 2, 3 (see (2.4)), so we will assume $n \geq 5$. Let $p \in IE_n$, $a_1, \ldots, a_{n-1}, b \in \mathrm{SJ}(X)$. Now

$$a_1 \ldots a_{n-1} pbp = (a_1 a_2 \ldots a_{n-1} p) bp \in \mathrm{SJ}(X)\mathrm{SJ}(X)\mathrm{SJ}(X) bp \subseteq \mathrm{SJ}(X)\mathrm{SJ}(X)\mathrm{SJ}(X)$$

since $p \in IE_n \subseteq IE_5$ and we have shown $U_p b = pbp \in IE_n$.

The proof of (1.6)(i) without taking traces applies here with obvious changes to show that IE_n is an outer ideal when n is odd. We know from (2.4) that $IE_1 = I_1$. We will show that for any even k, IE_{k+1} is an outer ideal of SJ(X). Let $p \in IE_{k+1}$, $a_1, \ldots, a_k, b \in \mathrm{SJ}(X)$. By (1.5), using $p \in IE_{k+1}$ and (2.5)(3), we obtain

$$\begin{aligned} &a_1 \ldots a_k bpb - a_1 \{a_2 \ldots a_k bp\} b - (-1)^{k/2} a_1 p a_{k-1} a_k \ldots a_3 a_4 b a_2 b \\ &\quad \in \mathrm{SJ}(X)\mathrm{SJ}(X)\mathrm{SJ}(X). \end{aligned}$$

Moreover

$$\begin{aligned} &a_1 \{a_2 \ldots a_k bp\} b \pm a_1 p a_{k-1} a_k \ldots a_3 a_4 b a_2 b \\ &\quad = a_1 \{a_2 \ldots a_k bp\} b \pm a_1 p a_{k-1} a_k \ldots a_3 a_4 (U_b a_2) \\ &\quad \in \mathrm{SJ}(X)\mathrm{SJ}(X)\mathrm{SJ}(X) \pm \overbrace{\mathrm{SJ}(X) p \mathrm{SJ}(X) \ldots \mathrm{SJ}(X)}^{k+1 \text{ factors}} \subseteq \mathrm{SJ}(X)\mathrm{SJ}(X)\mathrm{SJ}(X) \end{aligned}$$

by (2.5)(3) and $p \in IE_{k+1} \subseteq E_{k+1}$. So we have shown $U_b p = bpb \in IE_{k+1}$. This finishes the proof since SJ(X) is unital. □

The proofs of (1.9) and (1.10) without taking traces also apply for imbedded n-tad eaters, so that one readily obtains.

2.8. Theorem. *Let k, n be positive integers. Then*

(i) $2I_{4k} \subseteq IE_{4k+1}$, $2I_{4k+1} \subseteq IE_{4k+2}$ *and* $2I_{4k+2} \subseteq IE_{4k+3}$.

(ii) *If* $\frac{1}{2} \in \Phi$ *then* $I_{4k} = I_{4lk+1} = I_{4k+2} = I_{4k+3} = IE_{4k+1} = IE_{4k+2} = IE_{4k+3}$ *and* $IE_{n+1} \subseteq I_n$.

Note that (ii) also follows from (1.10) and (2.6) directly.

3. Hearty eaters

3.1. A (*unital*) *adic family* on the free special Jordan algebra SJ(X) is a family of n-linear maps $F_n : \mathrm{SJ}(X)^n \to V$ into some Φ-module V for all $n \geq 1$ having the *unital Jordan-alternating properties*

(AI) $F_n(\ldots, 1, \ldots) = F_{n-1}(\ldots, \ldots)$

(AII) $F_n(\dots, a, a, \dots) = F_{n-1}(\dots, a^2, \dots)$ (it is a consequence of (AI) and (AIII))
(AIII) $F_n(\dots, a, b, a, \dots) = F_{n-2}(\cdots, U_a b, \dots)$
(and therefore by linearization also)
(AII′) $F_n(\dots, a_1, a_2, \dots) + F_n(\dots, a_2, a_1, \dots) = F_{n-1}(\dots, \{a_1 a_2\}, \dots)$
(AIII′) $F_n(\dots, a_1, a_2, a_3, \dots) + F_n(\dots, a_3, a_2, a_1, \dots) = F_{n-2}(\dots, \{a_1 a_2 a_3\}, \dots)$.

In [4, p. 188], McCrimmon and Zelmanov give two more conditions called (AIV) and (AV) on compatibility of the maps F_n with tetrads and pentads. Our definition is just apparently more general. Indeed we will show below (see (3.7)) that, for adic families on SJ(X), (AIV) and (AV) are consequences of the previous axioms (AI)–(AIII).

3.2. The families of n-tads, imbedded n-tads and associative n-tads are clearly examples of the adic families on SJ(X) where $V = \mathrm{Ass}(X)$.

3.3. If $\mathscr{F}$ is a collection of adic families on SJ(X), we say that $p(y_1, \dots, y_m) \in \mathrm{SJ}(X)$ $(y_1, \dots, y_m \in X)$ *eats $\mathscr{F}$-n-tads* if

$$F_n(x_1, \dots, x_{n-1}, p) = \sum_{i=1}^{k} F_3(p_1^i, p_2^i, p_3^i) \in F_3(\mathrm{SJ}(X), \mathrm{SJ}(X), \mathrm{SJ}(X))$$

for some $p_j^i(x_1, \dots, x_{n-1}, y_1, \dots, y_m) \in \mathrm{SJ}(X)$, $x_1, \dots, x_{n-1} \in X \setminus \{y_1, \dots, y_m\}$, for any $\{F_n\}_{n \geq 1} \in \mathscr{F}$. When $\mathscr{F}$ consists of all possible adic families we will call such a p a *hearty n-tad eater*. The set $\mathscr{H}E_n$ of all hearty n-tad eaters is a submodule of SJ(X) whose core will be denoted by $\mathscr{H}_n$. Again, these are linearization-invariant ideals invariant under all endomorphisms and derivations of SJ(X) [4, p. 189].

Our aim is to show that all adic families can be reduced in a certain sense to the adic family of associative n-tads so that hearty n-tad eaters are just imbedded n-tad eaters.

3.4. Let $F = \{F_n\}_{n \geq 1}$ be an adic family on SJ(X) into V. We define a product in the direct sum of Φ-modules $\mathrm{Ass}(X) \oplus V$ by

$$(x_1 \dots x_n + v)(y_1 \dots y_m + w) = x_1 \dots x_n y_1 \dots y_m + F_{n+m}(x_1, \dots, x_n, y_1, \dots, y_m) \tag{1}$$

$$(1 + u)(x_1 \dots x_n + v) = (x_1 \dots x_n + v)(1 + u) = x_1 \dots x_n + F_n(x_1, \dots, x_n) \tag{2}$$

for any $x_1, \dots, x_n, y_1, \dots, y_m \in X$, extended to $\mathrm{Ass}(X) \oplus V$ by bilinearity. It is straightforward that the algebra obtained is associative. Let

$$\varphi_F : \mathrm{Ass}(X) \to \mathrm{Ass}(X) \oplus V$$

be the algebra homomorphism extending the map

$$X \to \mathrm{Ass}(X) \oplus V, \qquad x_i \mapsto x_i + F_1(x_i).$$

It is clear that φ_F is injective, so that the image $\varphi_F(\mathrm{Ass}(X))$, denoted by $\mathrm{Ass}_F(X)$, is a subalgebra of the one defined by (1) and (2), isomorphic to $\mathrm{Ass}(X)$. Namely, the algebra isomorphism

$$\varphi_F : \mathrm{Ass}(X) \to \mathrm{Ass}_F(X)$$

is given by

$$1 \mapsto 1 + F_1(1)$$

$$x_1 \ldots x_n \mapsto x_1 \ldots x_n + F_n(x_1, \ldots, x_n)$$

for any $x_1, \ldots, x_n \in X$.

3.5. Lemma. *Under the conditions of* (3.4)

$$\varphi_F(p_1 \ldots p_n) = p_1 \ldots p_n + F_n(p_1, \ldots, p_n) \in \mathrm{Ass}_F(X)$$

for any $p_1, \ldots, p_n \in \mathrm{SJ}(X)$.

Proof. By multilinearity of the product in $\mathrm{Ass}(X)$ and F_n we can assume that $p_1, \ldots, p_n$ are Jordan monomials (see (0.4)) different from 1. We proceed by induction on $m = (\sum_{i=1}^{n} \deg p_i) - n$. If $m = 0$ then $p_1, \ldots, p_n \in X$ and the result follows from the definition of φ_F. Let $m \geq 1$. Then there exists $j \in \{1, \ldots, n\}$ such that $\deg p_j \geq 2$. We must consider four different cases:

$$p_j = a^2, \ a \circ b, \ U_a b, \ U_{a,b} c,$$

where a, b, c are Jordan monomials and $\deg a, \deg b, \deg c < \deg p_j$. If $p_j = a^2$ then

$$\begin{aligned} p_1 \cdots p_n + F_n(p_1, \ldots, p_n) &= p_1 \ldots p_{j-1} a^2 p_{j+1} \ldots p_n + F_n(p_1, \ldots, p_{j-1}, a^2, p_{j+1}, \ldots, p_n) \\ &= p_1 \ldots p_{j-1} a a p_{j+1} \ldots p_n + F_{n+1}(p_1, \ldots, p_{j-1}, a, a, p_{j+1}, \ldots, p_n) \quad \text{(by (AII))} \\ &= \varphi_F(p_1 \ldots p_n) \in \mathrm{Ass}_F(X) \end{aligned}$$

by the induction assumption. The cases $p_j = a \circ b, U_a b, U_{a,b} c$ also follow from the induction assumption after applying (AII′), (AIII) and (AIII′), respectively. □

3.6. Theorem. *For any integer* $n \geq 1$

$$IE_n = \mathscr{H} E_n \quad \textit{and} \quad I_n = \mathscr{H}_n.$$

Proof. From (3.2), $\mathscr{H} E_n \subseteq IE_n$ and $\mathscr{H}_n \subseteq I_n$. Thus, let $p(y_1, \ldots, y_m) \in \mathrm{SJ}(X)$ $(y_1, \ldots, y_m \in X)$ be an imbedded n-tad eater. By (2.3) p eats associative n-tads, so that

$$x_1 \ldots x_{n-1} p = \sum_{i=1}^{k} q_1^i q_2^i q_3^i, \tag{1}$$

where $q_j^i(x_1, \ldots, x_{n-1}, y_1, \ldots, y_m) \in \mathrm{SJ}(X)$, $x_1, \ldots, x_{n-1} \in X \setminus \{y_1, \ldots, y_m\}$. Now, for any adic family F the isomorphism φ_F applied to (1) yields

$$x_1 \ldots x_{n-1} p + F_n(x_1, \ldots, x_{n-1}, p) = \sum_{i=1}^{k} q_1^i q_2^i q_3^i + \sum_{i=1}^{k} F_3(q_1^i, q_2^i, q_3^i)$$

using (3.5). Hence,

$$F_n(x_1, \ldots, x_{n-1}, p) = \sum_{i=1}^{k} F_3(q_1^i, q_2^i, q_3^i)$$

and p eats $F - n$-tads. We have shown $IE_n \subseteq \mathscr{H} E_n$, from which $I_n \subseteq \mathscr{H}_n$ is clear. □

3.7. Remarks. (i) Axioms (AIV) and (AV) of [4, 13.6] follow from (3.1) (AI)–(AIII). If $\{F_n\}_{n \geq 1}$ is an adic family on $\mathrm{SJ}(X)$ in the sense of (3.1) and $a_1, \ldots, a_k \in \mathrm{SJ}(X)$ satisfy $\{a_1 \ldots a_k\} \in \mathrm{SJ}(X)$ then

$$F_n(\ldots, a_1, \ldots, a_k, \ldots) + F_n(\ldots, a_k, \ldots, a_1, \ldots) = F_{n-(k-1)}(\ldots, \{a_1 \ldots a_k\}, \ldots) \tag{1}$$

using (3.5) when φ_F is applied to the equality

$$\ldots a_1 \ldots a_k \ldots + \ldots a_k \ldots a_1 \ldots = \ldots \{a_1 \ldots a_k\} \ldots$$

in $\mathrm{Ass}(X)$.

(ii) It is not true that for adic families on arbitrary special Jordan algebras the axioms (AIV) and (AV) on compatibility with tetrads and pentads are automatic. For example, let J and J' be special Jordan algebras, A and A' associative $*$-envelopes of J and J', respectively, and $f: J \to J'$ a Jordan homomorphism. We can consider the family $\{F_n : \mathrm{SJ}(X)^n \to A'\}_{n \geq 1}$ of n-linear maps defined by

$$F_n(x_1, \ldots, x_n) := f(x_1) \ldots f(x_n)$$

which have the Jordan-alternating properties

(AI) $F_n(\ldots, 1, \ldots) = F_{n-1}(\ldots, \ldots)$ (when J and J' are unital and f is unit-preserving),

(AII) $F_n(\ldots, x, x, \ldots) = F_{n-1}(\ldots, x^2, \ldots)$,

(AIII) $F_n(\ldots, x, y, x, \ldots) = F_{n-2}(\ldots, U_x y, \ldots)$.

But $\{F_n\}_{n \geq 1}$ is compatible with tetrads

(AIV) $F_n(\ldots, x_1, x_2, x_3, x_4, \ldots) + F_n(\ldots, x_4, x_3, x_2, x_1, \ldots) = F_{n-3}(\ldots, \{x_1 x_2 x_3 x_4\}, \ldots)$ whenever $\{x_1 x_2 x_3 x_4\} \in J$,

if and only if f preserves tetrads

$$f(\{x_1 x_2 x_3 x_4\}) = \{(f(x_1) f(x_2) f(x_3) f(x_4)\} \quad \text{whenever } \{x_1 x_2 x_3 x_4\} \in J.$$

Since examples of Jordan homomorphisms not preserving tetrads are known [3, Example in p. 459, Remark 1.7], the existence of families $\{F_n\}_{n \geq 1}$ not compatible with tetrads and satisfying the Jordan-alternating properties (3.1) follows.

(iii) However, there exists a large class of special Jordan algebras in which every family satisfying the Jordan-alternating properties is compatible with tetrads and pentads. Following [3] a special Jordan algebra J is a $\mathscr{Z}$-algebra if J equals $\mathscr{Z}(J)$ the ideal generated by all values $Z_{48}(a_1, \ldots, a_{12})$ for $a_i \in J$, where Z_{48} is a precise Zel'manov polynomial given in [4, pp. 192, 195] (see also [1, p. 182]). More generally, let J satisfy

$$J = E_5(J) \tag{1}$$

(in particular, any $\mathscr{Z}$-algebra satisfies the above equality since $\mathscr{Z} \subseteq E_5$ [4, 14.2]), A be a $*$-envelope for J, and $\{F_n\}_{n \geq 1}$ be a family of n-linear maps on J into a Φ-module V satisfying the Jordan-alternating properties (3.1). We note that for any map f from X into J, an adic family $\{F_n^f\}_{n \geq 1}$ on SJ(X) can be obtained by

$$F_n^f(p_1, \ldots, p_n) = F_n(\hat{f}(p_1), \ldots, \hat{f}(p_n)),$$

for $p_i \in \mathrm{SJ}(X)$, where $\hat{f}$ is the unique algebra homomorphism extension of f to SJ(X).

Now, fix $a_1, \ldots, a_k, b_1, b_2, b_3, b_4, d_1, \ldots, d_l$ in J for some $n = k + 4 + l$. By (1), there exists $p(z_1, \ldots, z_m)$ in the submodule E_5 of SJ(X), and $c_1, \ldots, c_m \in J$, such that $b_4 = p(c_1, \ldots, c_m)$. We choose pairwise different $x_1, \ldots, x_k, y_1, y_2, y_3, z_1, \ldots, z_m, t_1, \ldots, t_l \in X$ and consider the map $f: X \to J$ sending $x_i \mapsto a_i$ $(i = 1, \ldots, k)$, $y_i \mapsto b_i$ $(i = 1, 2, 3)$, $z_i \mapsto c_i$ $(i = 1, \ldots, m)$, $t_i \mapsto d_i$ $(i = 1, \ldots, l)$, and $u \mapsto 0$ for all $u \in X \setminus \{x_1, \ldots, x_k, y_1, y_2, y_3, z_1, \ldots, z_m, t_1, \ldots, t_l\}$. So, we have

$$\begin{aligned}
&F_n(a_1, \ldots, a_k, b_1, b_2, b_3, b_4, d_1, \ldots, d_l) + F_n(a_1, \ldots, a_k, b_4, b_3, b_2, b_1, d_1, \ldots, d_l)\\
&\quad = F_n(\hat{f}(x_1), \ldots, \hat{f}(x_k), \hat{f}(y_1), \hat{f}(y_2), \hat{f}(y_3), \hat{f}(p(z_1, \ldots, z_m)), \hat{f}(t_1), \ldots, \hat{f}(t_l))\\
&\qquad + F_n(\hat{f}(x_1), \ldots, \hat{f}(x_k), \hat{f}(p(z_1, \ldots, z_m)), \hat{f}(y_3), \hat{f}(y_2), \hat{f}(y_1), \hat{f}(t_1), \ldots, \hat{f}(t_l))\\
&\quad = F_n^f(x_1, \ldots, x_k, y_1, y_2, y_3, p(z_1, \ldots, z_m), t_1, \ldots, t_l)\\
&\qquad + F_n^f(x_1, \ldots, x_k, p(z_1, \ldots, z_m), y_3, y_2, y_1, t_1, \ldots, t_l)\\
&\quad = F_{n-3}^f(x_1, \ldots, x_k, \{y_1 y_2 y_3 p(z_1, \ldots, z_m)\}, t_1, \ldots, t_l)
\end{aligned}$$

($\{F_n^f\}$ is an adic family on SJ(X), hence it is compatible with tetrads, and the element $\{y_1 y_2 y_3 p(z_1, \ldots, z_m)\} \in \mathrm{SJ}(X)$ by (1))

$$\begin{aligned}
&\quad = F_{n-3}(\hat{f}(x_1), \ldots, \hat{f}(x_k), \hat{f}(\{y_1 y_2 y_3 p(z_1, \ldots, z_m)\}), \hat{f}(t_1), \ldots, \hat{f}(t_l))\\
&\quad = F_{n-3}(a_1, \ldots, a_k, \{b_1 b_2 b_3 b_4\}, d_1, \ldots, d_l),
\end{aligned}$$

where for the last equality we use the fact that if

$$\{y_1 y_2 y_3 p(z_1, \ldots, z_m)\} = q(y_1, y_2, y_3, z, \ldots, z_m) \in \mathrm{SJ}(X)$$

then

$$\begin{aligned}
\hat{f}(\{y_1 y_2 y_3 p(z_1, \ldots, z_m)\}) &= \hat{f}(q(y_1, y_2, y_3, z_1, \ldots, z_m))\\
&= q(f(y_1), f(y_2), f(y_3), f(z_1), \ldots, f(z_m))
\end{aligned}$$

$$= q(b_1, b_2, b_3, c_1, \ldots, c_m)$$
$$= \{b_1 b_2 b_3 p(c_1, \ldots, c_m)\} = \{b_1 b_2 b_3 b_4\} \in J.$$

Analogously, it is proved that $\{F_n\}$ is compatible with pentads.

(iv) The construction of $\mathrm{Ass}_F(X)$ provides a way to obtain any adic family $\{F_n\}_{n \geq 1}$ in $\mathrm{SJ}(X)$ from the adic family $\{A_n\}_{n \geq 1}$ consisting of the associative n-tads

$$A_n : \mathrm{SJ}(X)^n \to \mathrm{Ass}(X), \qquad A_n(a_1, \ldots, a_n) = a_1 \ldots a_n.$$

Indeed, the maps F_n, $n \geq 1$, are given by the compositions

$$F_n = \pi_V \circ \varphi_F \circ A_n,$$

where π_V denotes the canonical projection on V of $\mathrm{Ass}_F(X) \subseteq \mathrm{Ass}(X) \oplus V$.

3.8. Hearty Eaters for Jordan Triple Systems. The construction of the algebra $\mathrm{Ass}_F(X)$ given in (3.4) obviously can be generalized to triple systems so that one can show (formally with the same proofs) that hearty eaters in triples [1] are exactly the polynomials in the free special Jordan triple systems which eat associative n-tads (of odd length!). As in (3.7)(i), for an adic family on the free Jordan triple system $\mathrm{JTS}(X)$ condition (T2) in [1, 3.7] is a consequence of (T1).

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